

Title	Life-span of Classical Solutions to Nonlinear Wave Equations in Four Space Dimensions(Nonlinear Evolution Equations and Applications)
Author(s)	Li, Ta-tsien
Citation	数理解析研究所講究録 (1996), 966: 95-109
Issue Date	1996-09
URL	<a href="http://hdl.handle.net/2433/60604">http://hdl.handle.net/2433/60604</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

# Life-span of Classical Solutions to Nonlinear Wave Equations in Four Space Dimensions

Li Ta-tsien (Li Da-qian) \*

**Abstract** In this paper we prove that in four-space-dimensional case, L. Hörmander's estimate  $\tilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-1}\}$  ( $A > 0$ , constant) can be improved by  $\tilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-2}\}$  on the lower bound of the life-span  $\tilde{T}(\varepsilon)$  of classical solutions to the Cauchy problem with small initial data  $(u, u_t)(0, x) = \varepsilon(\phi(x), \psi(x))$  for nonlinear wave equations of the form  $\square u = F(u, Du, D_x Du)$ , where  $F(\hat{\lambda}) = O(|\hat{\lambda}|^2)$  in a neighbourhood of  $\hat{\lambda} = 0$ .

## 1 Introduction

Consider the Cauchy problem for fully nonlinear wave equations with small initial data:

$$\square u = F(u, Du, D_x Du), \quad (1.1)$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x), \quad (1.2)$$

where

$$\square = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \quad (1.3)$$

---

\*Department of Mathematics, Fudan University, Shanghai 200433, China.

is the wave operator,

$$D_x = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \quad D = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \quad (1.4)$$

$\phi, \psi \in C_0^\infty(\mathbb{R}^n)$  and  $\varepsilon > 0$  is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \dots, n; (\lambda_{ij}), i, j = 0, 1, \dots, n, i + j \geq 1). \quad (1.5)$$

Suppose that in a neighbourhood of  $\hat{\lambda} = 0$ , the nonlinear term  $F = F(\hat{\lambda})$  in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \quad (1.6)$$

where  $\alpha$  is an integer  $\geq 1$ .

For all integers  $n, \alpha$  with  $n \geq 1$  and  $\alpha \geq 1$ , the lower bound of the life-span of classical solutions to (1.1)-(1.2) was studied by S. Klainerman [1]-[2], [5], [7], J. L. Shatah [3], S. Klainerman & G. Ponce [4], F. John [6], F. John & S. Klainerman [8], M. Kovalyov [9], L. Hörmander [10], Li Ta-tsien & Yu Xin [11] etc. for the special case

$$F = F(Du, D_x Du),$$

and by D. Christodoulou [12], Li Ta-tsien & Chen Yun-mei [13], L. Hörmander [14], H. Lindblad [15], Li Ta-tsien & Yu Xin [16], Li Ta-tsien & Zhou Yi [17], Li Ta-tsien, Yu Xin & Zhou Yi [18]-[20], Li Ta-tsien & Zhou Yi [21]-[23] etc. for the general case

$$F = F(u, Du, D_x Du).$$

A summary of all the results mentioned above can be found in Li Ta-tsien & Chen Yu-mei [24]. All these lower bounds, except in the case  $n = 4, \alpha = 1$  and  $\partial_u^2 F(0, 0, 0) \neq 0$ , have been known to be sharp.

Thus, in the case  $n = 4$ ,  $\alpha = 1$  and  $\partial_u^2 F(0, 0, 0) \neq 0$ , it is natural to ask if the lower bound of the life-span

$$\tilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-1}\}, \quad (1.7)$$

where  $A$  is a positive constant independent of  $\varepsilon$ , originally obtained by L. Hörmander [14] and then, by means of the global iteration method, by Li Ta-tsien & Yu Xin [16], is sharp or not. In this paper, as a joint work with Zhou Yi, we shall prove that (1.7) can be improved by

$$\tilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-2}\}, \quad (1.8)$$

where  $A$  is a positive constant independent of  $\varepsilon$  (see Li Ta-tsien & Zhou Yi [35]-[36]).  $\square$

## 2 Motivation

By L. Hörmander [14] and Li Ta-tsien & Zhou Yi [17], if there is no  $u^2$  term in the Taylor expansion of  $F$ , i.e.,  $\partial_u^2 F(0, 0, 0) = 0$ , then in four space dimensions Cauchy problem (1.1)-(1.2) always admits a unique global classical solution on  $t \geq 0$ , provided that  $\varepsilon > 0$  is suitably small.

In order to illustrate the motivation of expecting estimate (1.8), as the ‘worst’ case of equation (1.1) we consider the equation

$$\square u = u^2 \quad (2.1)$$

which can be regarded as a special case ( $p = 2$ ) of the following equation

$$\square u = |u|^p \quad (p > 1). \quad (2.2)$$

When  $n = 3$ , F. John [25] proved that if  $p > 1 + \sqrt{2}$ , then for  $\varepsilon > 0$  suitably small, Cauchy problem (2.2) and (1.2) admits a unique global solution on  $t \geq 0$ ;

while, if  $1 < p < 1 + \sqrt{2}$ , then any nontrivial solution with compact support to equation (2.2) must blow up in a finite time. Thus, in the case  $n = 3$  the critical value of  $p$  is equal to  $p_0(3) = 1 + \sqrt{2}$ . In general, as suggested and studied by W. Strauss [26], R. T. Glassey [27]-[28], T. C. Sideris [29], J. Schaeffer [30] etc., in  $n$  space dimensions,  $p_0(n)$ , the critical value of  $p$ , should be the positive root of the following quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \quad (2.3)$$

In particular, we have  $p_0(4) = 2$ . Hence, equation (2.1) corresponds to the critical value of  $p$  in four space dimensions.

When  $p = p_0(n)$  with  $n = 2, 3$ , Zhou Yi [31]-[32] proved that the life-span  $\tilde{T}(\varepsilon)$  of solutions to Cauchy problem (2.2) and (1.2) satisfies

$$\tilde{T}(\varepsilon) \approx \exp\{A\varepsilon^{-p(p-1)}\}, \quad (2.4)$$

where  $A$  is a positive constant independent of  $\varepsilon$ ; namely, there exist two positive constants  $A_1$  and  $A_2$  independent of  $\varepsilon$ , such that

$$\exp\{A_1\varepsilon^{-p(p-1)}\} \leq \tilde{T}(\varepsilon) \leq \exp\{A_2\varepsilon^{-p(p-1)}\}.$$

We guess that (2.4) still holds in the case  $n = 4$ . If so, in four space dimensions the life-span of solutions to Cauchy problem (2.2) and (1.2) should satisfy

$$\tilde{T}(\varepsilon) \approx \exp\{A\varepsilon^{-2}\}. \quad (2.5)$$

This consideration leads us to prove (1.8) for Cauchy problem (1.1)-(1.2) and to believe that this lower bound of the life-span should be sharp.  $\square$

### 3 Proof of the main result

The general framework which we shall use to prove (1.8) is still the global iteration method, suggested in Li Ta-tsien & Chen Yun-mei [13] and Li Ta-tsien & Yu Xin

[16].

First of all, just by differentiation, it suffices to consider the Cauchy problem for the following general kind of quasilinear wave equations:

$$\square u = \sum_{i,j=1}^n b_{ij}(u, Du)u_{x_i x_j} + 2 \sum_{j=1}^n a_{0j}(u, Du)u_{tx_j} + F_0(u, Du), \quad (3.1)$$

$$t = 0 : u = \varepsilon \phi(x), u_t = \varepsilon \psi(x), \quad (3.2)$$

where  $b_{ij}$ ,  $a_{0j}$  ( $i, j = 1, \dots, n$ ) and  $F_0$  satisfy certain suitable assumptions. Without loss of generality, in what follows we may suppose that

$$\text{supp } \{\phi, \psi\} \subseteq \{x \mid |x| \leq 1\}. \quad (3.3)$$

The solution  $u$  to Cauchy problem (3.1)-(3.2) (in which  $n = 4$ ) can be written as

$$u = w + u_\varepsilon, \quad (3.4)$$

where  $u_\varepsilon$  is the solution to the following Cauchy problem for the homogeneous wave equation:

$$\square u_\varepsilon = 0, \quad (3.5)$$

$$t = 0 : u_\varepsilon = \varepsilon \phi(x), (u_\varepsilon)_t = \varepsilon \psi(x); \quad (3.6)$$

while  $w$  is the solution to the following Cauchy problem:

$$\square w = \sum_{i,j=1}^4 b_{ij}(u, Du)u_{x_i x_j} + 2 \sum_{j=1}^4 a_{0j}(u, Du)u_{tx_j} + F_0(u, Du), \quad (3.7)$$

$$t = 0 : w = 0, w_t = 0. \quad (3.8)$$

The iteration scheme is given after the preceding translation by

$$\left\{ \begin{array}{l} \square w = \sum_{i,j=1}^4 b_{ij}(u_\varepsilon + \omega, D(u_\varepsilon + \omega))(u_\varepsilon + \omega)_{x_i x_j} \\ \quad + 2 \sum_{j=1}^4 a_{0j}(u_\varepsilon + \omega, D(u_\varepsilon + \omega))(u_\varepsilon + \omega)_{tx_j} \\ \quad + F_0(u_\varepsilon + \omega, D(u_\varepsilon + \omega)), \\ t = 0 : w = 0, w_t = 0, \end{array} \right. \quad (3.9)$$

which defines a map

$$M : \omega \rightarrow w = M\omega. \square \quad (3.10)$$

For any integer  $N \geq 0$ , define

$$\|u(t, \cdot)\|_{\Gamma, N, p, q, \chi} = \sum_{|k| \leq N} \|\chi(t, \cdot) \Gamma^k u(t, \cdot)\|_{L^{p, q}(\mathbb{R}^n)}, \quad (3.11)$$

where  $\Gamma$  denotes the following set of partial differential operators:

$$\Gamma = \{D, L, \Omega\} \triangleq (\Gamma_1, \dots, \Gamma_\sigma), \quad (3.12)$$

in which

$$D = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \quad (3.13)$$

$$L = (L_a, a = 0, 1, \dots, n), \quad (3.14)$$

$$\Omega = (\Omega_{ij}, i, j = 1, \dots, n) \quad (3.15)$$

with

$$\begin{cases} L_0 = t\partial_t + x_1\partial_{x_1} + \dots + x_n\partial_{x_n}, \\ L_i = t\partial_{x_i} + x_i\partial_t, & i = 1, \dots, n, \\ \Omega_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}, & i, j = 1, \dots, n, \end{cases} \quad (3.16)$$

$1 \leq p, q \leq +\infty$ ,  $\chi(t, x)$  is the characteristic function of any given set in  $\mathbb{R}_+ \times \mathbb{R}^n$ ,  $k = (k_1, \dots, k_\sigma)$  are multi-indices and  $L^{p, q}(\mathbb{R}^n)$  is a function space, introduced by Li Ta-tsien & Yu Xin in [16], with the norm

$$\|f(\cdot)\|_{L^{p, q}(\mathbb{R}^n)} = \|f(r\xi)r^{\frac{n-1}{p}}\|_{L^p(0, +\infty; L^q(S^{n-1}))}, \quad (3.17)$$

where  $r = |x|$  and  $\xi \in S^{n-1}$ ,  $S^{n-1}$  being the unit sphere in  $\mathbb{R}^n$ . We write

$$\|u(t, \cdot)\|_{\Gamma, N, p, q, \chi} = \begin{cases} \|u(t, \cdot)\|_{\Gamma, N, p, \chi}, & \text{if } p = q, \\ \|u(t, \cdot)\|_{\Gamma, N, p, q}, & \text{if } \chi \equiv 1, \\ \|u(t, \cdot)\|_{p, q, \chi}, & \text{if } N = 0. \square \end{cases} \quad (3.18)$$

The essential point for the global iteration method is to choose a suitable function space with which we shall work in the iteration. This space should

simultaneously reflect the decay property and the energy estimate of solutions to the linear wave equation.

In the present situation, the space is chosen as follows:

$$X_{S,E,T} = \{\omega(t, x) \mid D_{S,T}(\omega) \leq E, \partial_t^l \omega(0, x) = w_l^{(0)}(x) \ (l = 0, 1, \dots, S+1)\}, \quad (3.19)$$

where  $S$  is an integer  $\geq 11$ ,  $E$  and  $T$  are positive numbers and

$$\begin{aligned} D_{S,T}(\omega) = & \sum_{|i|=1}^2 \sup_{0 \leq t \leq T} \|D^i \omega(t, \cdot)\|_{\Gamma, S, 2} + \sup_{0 \leq t \leq T} \|\omega(t, \cdot)\|_{\Gamma, S, 2, \chi_1} \\ & + \sup_{0 \leq t \leq T} (1+t) \|\omega(t, \cdot)\|_{\Gamma, S, 6, 2, \chi_2} + \sup_{0 \leq t \leq T} (1+t)^{-\frac{3}{2}} (\ln(2+t))^{-1} |||\omega(t, \cdot)||| \\ & + \sup_{\substack{0 \leq t \leq T \\ x \in \mathbf{R}^4}} (1+t)^{\frac{3}{2}} (1+|t-|x||)^{\frac{1}{2}} \sum_{|k| \leq S-3} |\Gamma^k \omega(t, x)|, \end{aligned} \quad (3.20)$$

in which  $\chi_1$  is the characteristic function of the set  $\{(t, x) \mid |x| \leq \frac{t+1}{2}\}$ ,  $\chi_2 = 1 - \chi_1$  and

$$|||\omega(t, \cdot)||| = \sum_{|k| \leq S} \|(1+|t-|\cdot||)^{-\frac{1}{2}} \chi_2 \Gamma^k \omega(t, \cdot)\|_{L^{1,2}(\mathbf{R}^4)}. \quad (3.21)$$

Moreover,  $w_0^{(0)} = w_1^{(0)} = 0$  and  $w_l^{(0)}(x)$  ( $l = 2, \dots, S+1$ ) are the values of  $\partial_t^l w(t, x)$  at  $t = 0$  formally determined by Cauchy problem (3.7)-(3.8) with (3.4)-(3.6).

Endowed with the metric

$$\rho(\bar{\omega}, \bar{\bar{\omega}}) = D_{S,T}(\bar{\omega} - \bar{\bar{\omega}}), \quad \forall \bar{\omega}, \bar{\bar{\omega}} \in X_{S,E,T}, \quad (3.22)$$

$X_{S,E,T}$  is a nonempty complete metric space, provided that  $\varepsilon > 0$  is suitably small.

Let  $\widetilde{X}_{S,E,T}$  be the subset of  $X_{S,E,T}$  composed of all elements  $\omega \in X_{S,E,T}$  such that

$$\text{supp } \omega \subseteq \{(t, x) \mid |x| \leq t+1\}. \quad (3.23)$$

If we can show that for  $\varepsilon > 0$  suitably small there exist  $E = E(\varepsilon)$  and  $T = T(\varepsilon)$  such that the map  $M$  has a unique fixed point  $w \in \widetilde{X}_{S,E(\varepsilon),T(\varepsilon)}$ , which implies that



$u = w + u_\varepsilon$  is the unique classical solution to Cauchy problem (3.1)-(3.2), then we get the following lower bound of the life-span

$$\tilde{T}(\varepsilon) \geq T(\varepsilon). \square \quad (3.24)$$

Noting that the initial data in (3.9) are zero and the nonlinear term in (3.7) is quadratic with respect to  $u = u_\varepsilon + w$ , we have

**Lemma 3.1.** For  $w = M\omega$ ,  $\partial_t^l w(0, x)$  ( $l = 0, 1, \dots, S+2$ ) are independent of  $\omega \in \widetilde{X}_{S,E,T}$ , and

$$\|w(0, \cdot)\|_{\Gamma, S+2, p, q} \leq C\varepsilon^2, \quad (3.25)$$

where  $1 \leq p, q \leq +\infty$ ,  $C$  is a positive constant independent of  $\varepsilon$  and  $\|w(0, \cdot)\|_{\Gamma, S+2, p, q}$  is the value of  $\|w(t, \cdot)\|_{\Gamma, S+2, p, q}$  at  $t = 0$ .  $\square$

According to the basic procedure of the global iteration method, in order to get the desired result, it suffices to show the following two lemmas.

**Lemma 3.2.** For any  $\omega \in \widetilde{X}_{S,E,T}$ ,  $w = M\omega$  satisfies

$$D_{S,T}(w) \leq C_1\{\varepsilon^2 + (R + \sqrt{R})(E + D_{S,T}(w))\}. \square \quad (3.26)$$

**Lemma 3.3.** Let  $\bar{\omega}, \bar{\bar{\omega}} \in \widetilde{X}_{S,E,T}$ . If  $\bar{w} = M\bar{\omega}$  and  $\bar{\bar{w}} = M\bar{\bar{\omega}}$  also satisfy  $\bar{w}, \bar{\bar{w}} \in \widetilde{X}_{S,E,T}$ , then

$$D_{S-1,T}(\bar{w} - \bar{\bar{w}}) \leq C_2(R + \sqrt{R})(D_{S-1,T}(\bar{w} - \bar{\bar{w}}) + D_{S-1,T}(\bar{\omega} - \bar{\bar{\omega}})). \square \quad (3.27)$$

In Lemma 3.2 and Lemma 3.3,  $C_1, C_2$  are positive constants and

$$R = R(\varepsilon, E, T) = E \ln(2 + T) + \varepsilon \sqrt{\ln(2 + T)}. \quad (3.28)$$

Based on these two lemmas, the standard contraction mapping principle can be easily used to show that the existence time interval  $[0, T(\varepsilon)]$  will be determined by

$$R(\varepsilon, E(\varepsilon), T(\varepsilon)) + \sqrt{R(\varepsilon, E(\varepsilon), T(\varepsilon))} \leq \frac{1}{C_0}, \quad (3.29)$$

where  $C_0 = 3 \max(C_1, C_2)$  and  $E(\varepsilon) = C_0 \varepsilon^2$ . Obviously, if we take

$$T(\varepsilon) = \exp\{\bar{A}\varepsilon^{-2}\} - 2 \quad (3.30)$$

with  $\bar{A} > 0$  suitably small, then (3.29) holds. This gives the desired estimate (1.8).  $\square$

For the proof of Lemmas 3.2 and 3.3, we need some refined estimates on the solution to the Cauchy problem

$$\begin{cases} \square u = F(t, x), \\ t = 0 : u = f(x), u_t = g(x) \end{cases} \quad (3.31)$$

in four space dimensions.

As mentioned above, L. Hörmander's estimate (1.7) was reproved by Li Tatsien & Yu Xin in [16]. The key tool in the proof is the following lemma.

**Lemma 3.4.** Suppose that  $n \geq 3$ . Let  $u = u(t, x)$  be the solution to Cauchy problem (3.31). Then

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\leq C\{\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^q(\mathbb{R}^n)} \\ &\quad + \int_0^t [\|\chi_1 F(\tau, \cdot)\|_{L^q(\mathbb{R}^n)} + (1 + \tau)^{-\frac{n-2}{2}} \|\chi_2 F(\tau, \cdot)\|_{L^{1,2}(\mathbb{R}^n)}] d\tau, \\ &\quad \forall t \geq 0, \end{aligned} \quad (3.32)$$

where  $\frac{1}{q} = \frac{1}{2} + \frac{1}{n}$ ,  $\chi_1$  is the characteristic function of the set  $\{(t, x) \mid |x| \leq \frac{1+t}{2}\}$ ,  $\chi_2 = 1 - \chi_1$  and  $C$  is a positive constant.  $\square$

Inequality (3.32), established in Li Ta-tsien & Yu Xin [16], may be regarded as an improved form of Von Wahl's inequality (cf. [33]).

Based on a Sobolev embedding theorem in the radial direction, the idea of the proof of Lemma 3.4 can be applied to get the following two lemmas.

**Lemma 3.5.** Suppose that  $n = 4$ . Let  $u = u(t, x)$  be the solution to Cauchy problem (3.31) with

$$\begin{cases} \text{supp } \{f, g\} \subseteq \{x \mid |x| \leq 1\}, \\ \text{supp } F \subseteq \{(t, x) \mid |x| \leq t + 1\}. \end{cases} \quad (3.33)$$

Then

$$\begin{aligned} \|\chi_1 u(t, \cdot)\|_{L^2(\mathbb{R}^4)} &\leq C(1+t)^{-1} \{\|f\|_{L^2(\mathbb{R}^4)} + \|g\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} \\ &\quad + \int_0^t [(1+\tau)\|\chi_1 F(\tau, \cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} + \|\chi_2 F(\tau, \cdot)\|_{L^{1,2}(\mathbb{R}^4)}] d\tau\}, \\ &\quad \forall t \geq 0, \end{aligned} \quad (3.34)$$

where  $\chi_1$  is the characteristic function of the set  $\{(t, x) \mid |x| \leq \frac{t+1}{2}\}$ ,  $\chi_2 = 1 - \chi_1$  and  $C$  is a positive constant.  $\square$

**Lemma 3.6.** Suppose that  $n = 4$ . Let  $u = u(t, x)$  be the solution to Cauchy problem (3.31). Then

$$\begin{aligned} \|\chi_2 u(t, \cdot)\|_{L^{p,2}(\mathbb{R}^4)} &\leq C(1+t)^{-3(\frac{1}{2}-\frac{1}{p})} \{\|f\|_{\dot{H}^{S_0}(\mathbb{R}^4)} + \|g\|_{L^\gamma(\mathbb{R}^4)} \\ &\quad + \int_0^t [\|\chi_1 F(\tau, \cdot)\|_{L^\gamma(\mathbb{R}^4)} + (1+\tau)^{-(1+S_0)} \|\chi_2 F(\tau, \cdot)\|_{L^{1,2}(\mathbb{R}^4)}] d\tau\}, \\ &\quad \forall t \geq 0, \end{aligned} \quad (3.35)$$

where  $p > 2$ ,  $S_0 = \frac{1}{2} - \frac{1}{p}$ ,  $\frac{1}{\gamma} = \frac{1}{2} + \frac{1-S_0}{4}$ ,  $C$  is a positive constant and  $\dot{H}^{S_0}(\mathbb{R}^4)$  stands for the homogeneous Sobolev space equipped with the norm

$$\|f\|_{\dot{H}^{S_0}(\mathbb{R}^4)} = \| |\xi|^{S_0} \hat{f}(\xi) \|_{L^2(\mathbb{R}^4)}, \quad (3.36)$$

where  $\hat{f}(\xi)$  is the Fourier transformation of  $f(x)$ .  $\square$

Moreover, noting that the initial data have compact support, a version of Huyghen's principle can be used to improve Hörmander's  $L^1$ - $L^\infty$  estimate (see L. Hörmander [34]) and Lemma 3.4 as presented in Lemma 3.7 and Lemma 3.8 respectively.

**Lemma 3.7.** Under the assumptions of Lemma 3.5, we have

$$\begin{aligned} |u(t, x)| \leq & C(1 + t + |x|)^{-\frac{3}{2}}(1 + |t - |x||)^{-\frac{3}{2}} \{ \|u(0, \cdot)\|_{\Gamma, 4, 1} \\ & + \|u_t(0, \cdot)\|_{\Gamma, 3, 1} + (1 + |t - |x||) \sup_{0 \leq \tau \leq t} \|F(\tau, \cdot)\|_{\Gamma, 3, 1} \}, \\ & \forall t \geq 0, \forall x \in \mathbb{R}^4, \end{aligned} \quad (3.37)$$

where  $C$  is a positive constant.  $\square$

**Lemma 3.8.** Under the assumptions of Lemma 3.5, we have

$$\begin{aligned} & \|(1 + |t - |\cdot||)^{-\frac{1}{2}} \chi_2 u(t, x)\|_{L^{1,2}(\mathbb{R}^4)} \\ & \leq C(1 + t)^{\frac{3}{2}} \{ \|u(0, \cdot)\|_{\Gamma, 4, 1} + \|u_t(0, \cdot)\|_{\Gamma, 3, 1} \\ & \quad + \int_0^t [(1 + \tau)^{\frac{1}{3}} \|\chi_1 F(\tau, \cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^4)} + (1 + \tau)^{-1} \|\chi_2 F(\tau, \cdot)\|_{L^{1,2}(\mathbb{R}^4)}] d\tau \}, \\ & \quad \forall t \geq 0, \end{aligned} \quad (3.38)$$

where  $C$  is a positive constant.  $\square$

Lemmas 3.5-3.8 play an important role in the proof of Lemmas 3.2 and 3.3.  $\square$

## References

- [1] S. Klainerman, Global existence for nonlinear wave equations, Comm. Pure Appl. Math., 33(1980), 43-101.

- [2] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations, *Arch. Rat. Mech. Anal.*, 78(1982), 73-98.
- [3] J. L. Shatah, Global existence of small solutions to nonlinear evolution equations, *J. Diff. Equations*, 46(1982), 409-425.
- [4] S. Klainerman & G. Ponce, Global, small amplitude solutions to nonlinear evolution equations, *Comm. Pure Appl. Math.*, 36(1983), 133-141.
- [5] S. Klainerman, Uniform decay estimates and the Lorentz invariance of the classical wave equations, *Comm. Pure Appl. Math.*, 38(1985), 321-332.
- [6] F. John, Blow-up for quasilinear wave equations in three space dimensions, *Comm. Pure Appl. Math.*, 34(1981), 29-51.
- [7] S. Klainerman, On "almost global" solutions to quasilinear wave equations in three space dimensions, *Comm. Pure Appl. Math.*, 36(1983), 325-344.
- [8] F. John & S. Klainerman, Almost global existence to nonlinear wave equations in three space dimensions, *Comm. Pure Appl. Math.*, 37(1984), 443-455.
- [9] M. Kovalyov, Long-time behavior of solutions of a system of nonlinear wave equations, *Commu. in Partial Differential Equations*, 12(1987), 471-501.
- [10] L. Hörmander, The life span of classical solutions of nonlinear hyperbolic equations, *Institut Mittag-Leffler, Report No.5*, 1985.
- [11] Li Ta-tsien & Yu Xin, Durée de vie des solutions régulières pour les équations des ondes non linéaires, *C. R. Acad. Sci., Paris, t.309, Série I* (1989), 469-472;  
Life-span of classical solutions to fully nonlinear wave equations, *IMA Preprint Series #529*, June 1989.

- [12] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.*, 39(1986), 267-282.
- [13] Li Ta-tsien and Chen Yun-mei, Initial value problems for nonlinear wave equations, *Commu. Partial Differential Equations*, 13(1988), 383-422.
- [14] L. Hörmander, On the fully non-linear Cauchy problem with small data II, *Microlocal Analysis and Nonlinear Waves* (eds. M. Beals, R. Melrose and J. Rauch), Vol.30, IMA Volumes in Mathematics and its Applications, Springer-Verlag, Berlin, 1991, pp.51-81.
- [15] H. Lindblad, On the lifespan of solutions of nonlinear wave equations with small initial data, *Comm. Pure Appl. Math.*, 43(1990), 445-472.
- [16] Li Ta-tsien and Yu Xin, Life-span of classical solutions to fully nonlinear wave equations, *Commu. Partial Differential Equations*, 16(1991), 909-940.
- [17] Li Ta-tsien and Zhou Yi, Life-span of classical solutions to fully nonlinear wave equations II, *Nonlinear Analysis, Theory, Methods & Applications*, 19(1992), 833-853.
- [18] Li Ta-tsien, Yu Xin and Zhou Yi, Durée de vie des solutions régulières pour les équations des ondes non linéaires unidimensionnelles, *C. R. Acad. Sci., Paris*, 312, Série I (1991), 103-105.
- [19] Li Ta-tsien, Yu Xin and Zhou Yi, Life-span of classical solutions to one dimensional nonlinear wave equations, *Chin. Ann. of Math.*, 13B(1992), 266-279.

- [20] Li Ta-tsien, Yu Xin and Zhou Yi, Problème de Cauchy pour les équations des ondes non linéaires avec petites données initiales, C. R. Acad. Sci., Paris, 312, Série I (1991), 337-340.
- [21] Li Ta-tsien and Zhou Yi, Life-span of classical solutions to nonlinear wave equations in two space dimensions, J. Math. Pure et Appl., 73(1994), 223-249.
- [22] Li Ta-tsien and Zhou Yi, Life-span of classical solutions to nonlinear wave equations in two space dimensions II, J. Partial Differential Equations, 6(1993), 17-38.
- [23] Li Ta-tsien and Zhou Yi, Nonlinear stability for two space dimensional wave equations with higher order perturbation, Nonlinear World, 1(1994), 35-58.
- [24] Li Ta-tsien and Chen Yun-mei, *Global Classical Solutions for Nonlinear Evolution Equations*, Pitman Monographs and Surveys in Pure and Applied Mathematics 45, Longman Scientific & Technical, 1992.
- [25] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manus. Math., 28(1979), 235-268.
- [26] W. Strauss, Nonlinear scattering theory at low energy, J. Func. Anal., 41(1981), 110-133.
- [27] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, Math. Z., 177(1981), 323-340.
- [28] R. T. Glassey, Existence in the large for  $\square u = F(u)$  in two space dimensions, Math. Z., 178(1981), 223-261.

- [29] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Diff. Equations*, 52(1984), 378-406.
- [30] J. Schaeffer, The equation  $u_{tt} - \Delta u = |u|^p$  for the critical value of  $p$ , *Proc. Royal Soc. Edinburgh*, 101A(1985), 31-44.
- [31] Zhou Yi, Life span of classical solutions to  $\square u = |u|^p$  in two space dimensions, *Chin. Ann. of Math.*, 14B(1993), 225-236.
- [32] Zhou Yi, Blow up of classical solutions to  $\square u = |u|^p$  in three space dimensions, *J. Partial Differential Equations*, 5(1992), 21-32.
- [33] W. Von Wahl, Über die klassische lösbarkeit des Cauchy-Problems für nicht-lineare Wellengleichungen bei kleinen Anfangswerten und das asymptotische Verhalten der Lösungen, *Math. Z.*, 114(1970), 281-299.
- [34] L. Hörmander,  $L^1$ ,  $L^\infty$  estimates for the wave operator, *Analyse Mathématique et Applications*, Gauthier-Villars, Paris, 1988, pp.211-234.
- [35] Li Ta-tsien & Zhou Yi, Durée de vie des solutions régulières pour les équations des ondes non linéaires en dimension quatre d'espace, *C. R. Acad. Sc.*, Paris, 320, Série I (1995), 41-44.
- [36] Li Ta-tsien & Zhou Yi, A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions, to appear in *Indiana University Mathematics Journal*.